PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

LECTURE 9: PROJECTIVE GEOMETRY

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Have you ever wondered what it is like to move around inside a painting? It's a fun and evocative exercise in the imagination, and it was something I remember thinking about often as a child. However, as a child, I was ill-equipped to understand the geometry of the situation, or even what geometry means in this case.

In today's lecture, we'll be exploring this two-dimensional *projective geometry* and its higher-dimensional analogues. In outline, our plan is the following:

- Explain what geometry means for a painting
- Verify that the geometry is parabolic
- Describe how to move around within projective geometry
- Discuss what geodesics in the geometry look like

By the end of the lecture, we should have a decent idea of what it's like to move around inside the geometry of a painting. In particular, we'll have a better picture of what parabolic model geometries look like; we will further supplement this picture next time, when we talk about conformal geometry.

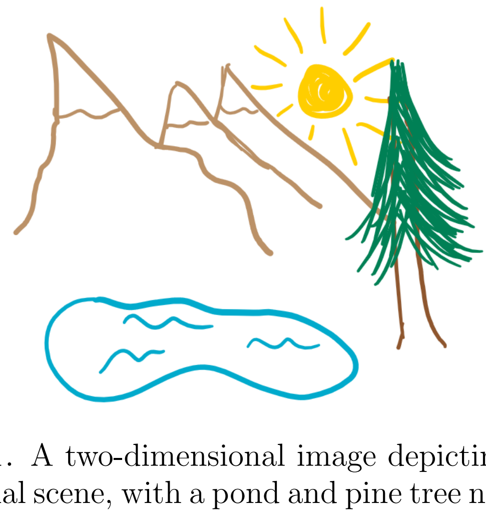

FIGURE 1. A two-dimensional image depicting a three-dimensional scene, with a pond and pine tree next to each other and a mountain range in the background, behind which is a setting sun

1. The geometry of a painting

Let's imagine a landscape painting: there is a pond with a small pine tree next to it, and behind these is a majestic mountain range, beyond which is a setting sun.

Our use of the words "behind" and "beyond" here suggests that, while the painting itself is two-dimensional, we think of the scene depicted as occurring in three dimensions. How do we get the two-dimensional image from the three-dimensional scene?

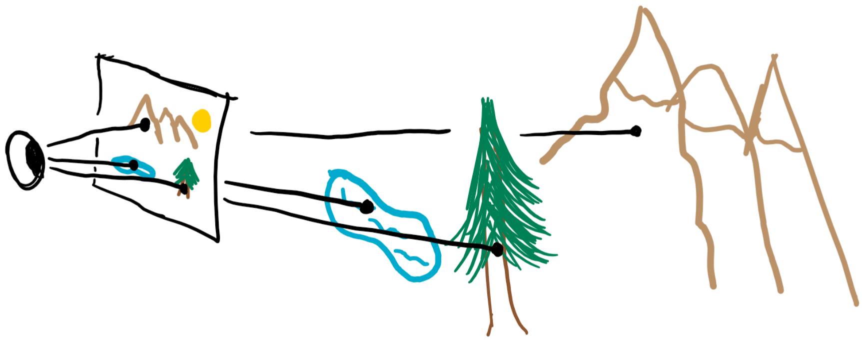

Figure 2. Each point of the canvas corresponds to a sight-line between the eye of the painter and a point in the scene

Let's suppose that the landscape occurs inside of $\mathbb{R}^3$, and that the painter observes the scene through their eye, which we place at the origin in $\mathbb{R}^3$. For each point $x \in \mathbb{R}^3$ in the scene, there is a unique line $\langle x \rangle = \mathbb{R}x$ through the origin that also contains this point; we call this line the *sight-line* through x. When the painter commits this scene to their canvas, they are identifying each point of their canvas with a corresponding sight-line, effectively *projecting* the three-dimensional scene down to a two-dimensional image.

Since the geometry comes from these sight-lines, its symmetries will be those that preserve them. A transformation that preserves lines in $\mathbb{R}^3$ *and* preserves the origin (where the eye is) is going to be an element of $\mathrm{GL}_3\,\mathbb{R}$. However, since the sight-lines are what we're really interested in and the center $\mathrm{Z}(\mathrm{GL}_3\,\mathbb{R}) = \mathbb{R}^\times \mathbb{1}$ sends each line through 0 to itself, we want to ignore these central elements. Thus, the model group of this geometry is $\mathrm{GL}_3\,\mathbb{R}/\mathbb{R}^\times\mathbb{1} = \mathrm{PGL}_3\,\mathbb{R}$.

The model group $\mathrm{PGL}_3\,\mathbb{R}$ acts transitively on the projective plane $\mathbb{RP}^2$, also known as the space of sight-lines in $\mathbb{R}^3$. Thus, defining P to be the stabilizer of the sight-line through $\left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]^\top$, we get a bijection between $\mathrm{PGL}_3\,\mathbb{R}/P$ and $\mathbb{RP}^2$. In short, our model for 2-dimensional projective geometry is $(\mathrm{PGL}_3\,\mathbb{R}, P)$.

More generally, we can consider m-dimensional projective geometry, which corresponds to the geometry of sight-lines inside of $\mathbb{R}^{m+1}$. In that case, our model is $(\mathrm{PGL}_{m+1}\,\mathbb{R}, P)$, where

$$P := \left\{ \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \in \mathrm{PGL}_{m+1}\,\mathbb{R} : a \in \mathbb{R}^\times, \alpha^\top \in \mathbb{R}^m, A \in \mathrm{GL}_m\,\mathbb{R} \right\}$$

is, again, the stabilizer of $\left[\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right]^\top$.

2. Parabolicity of projective geometry

The Killing form on $\mathfrak{pgl}_{m+1}\mathbb{R} := \mathfrak{gl}_{m+1}\mathbb{R}/\mathbb{R}\mathbb{1}$, where by definition elements of $\mathfrak{pgl}_{m+1}\mathbb{R}$ are equivalent if and only if they differ by a scalar multiple of the identity matrix, is given by

$$\mathfrak{B}\Big(\left(\begin{smallmatrix} -\mathrm{tr}(R) & \alpha \\ v & R \end{smallmatrix}\right), \left(\begin{smallmatrix} -\mathrm{tr}(S) & \beta \\ w & S \end{smallmatrix}\right) \Big) = 2(m+1)\Big(\mathrm{tr}(R)\mathrm{tr}(S) + \mathrm{tr}(RS) + \alpha(w) + \beta(v)\Big),$$

so

$$\mathfrak{B}\Big(\left(\begin{smallmatrix} -\mathrm{tr}(R) & \alpha \\ 0 & R \end{smallmatrix}\right), \left(\begin{smallmatrix} -\mathrm{tr}(S) & \beta \\ w & S \end{smallmatrix}\right) \Big) = 2(m+1)\Big(\mathrm{tr}(R)\mathrm{tr}(S) + \mathrm{tr}(RS) + \alpha(w)\Big),$$

which vanishes for all $\left(\begin{smallmatrix} -\mathrm{tr}(R) & \alpha \\ 0 & R \end{smallmatrix}\right) \in \mathfrak{p}$ precisely when $S = 0$ and $w = 0$. Thus, $\mathfrak{p}^\perp$ is the abelian subalgebra $\{\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right) : \alpha^\top \in \mathbb{R}^m\}$, hence $\mathfrak{p}$ is parabolic.

Choosing our Cartan involution θ to be given by $X \mapsto -X^\top$, so that

$$\theta\Big(\left(\begin{smallmatrix} r & \alpha \\ v & R \end{smallmatrix}\right) \Big) = \left(\begin{smallmatrix} -r & -v^\top \\ -\alpha^\top & -R^\top \end{smallmatrix}\right),$$

we get a grading of $\mathfrak{pgl}_{m+1}\mathbb{R}$ given by

$$\mathfrak{g}_{-1} = \mathfrak{g}_- := \{\left(\begin{smallmatrix} 0 & 0 \\ v & 0 \end{smallmatrix}\right) \in \mathfrak{pgl}_{m+1}\mathbb{R} : v \in \mathbb{R}^m\},$$

$$\begin{aligned}\mathfrak{g}_0 &:= \{\left(\begin{smallmatrix} r & 0 \\ 0 & R \end{smallmatrix}\right) \in \mathfrak{pgl}_{m+1}\mathbb{R} : r \in \mathbb{R},\ R \in \mathfrak{gl}_m\mathbb{R}\} \\ &= \left\{\left(\begin{smallmatrix} -\mathrm{tr}(S) & 0 \\ 0 & S \end{smallmatrix}\right) \in \mathfrak{pgl}_{m+1}\mathbb{R} : S \in \mathfrak{gl}_m\mathbb{R}\right\},\end{aligned}$$

and

$$\mathfrak{g}_1 = \mathfrak{p}_+ := \left\{\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{pgl}_{m+1}\mathbb{R} : \alpha^\top \in \mathbb{R}^m\right\}.$$

It is worth drawing attention to the fact that $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are abelian in this case, so that we only have three grading components. The horospherical subgroups

$$G_- := \{\left(\begin{smallmatrix} 1 & 0 \\ v & \mathbb{1} \end{smallmatrix}\right) \in \mathrm{PGL}_{m+1}\,\mathbb{R} : v \in \mathbb{R}^m\}$$

and

$$P_+ := \left\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & \mathbb{1} \end{smallmatrix}\right) \in \mathrm{PGL}_{m+1}\,\mathbb{R} : \alpha^\top \in \mathbb{R}^m\right\}$$

are, in particular, also abelian.

The grading element for this grading is given by

$$E_{\mathrm{gr}} := \tfrac{1}{m+1}\left(\begin{smallmatrix} m & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right) = \tfrac{1}{m+1}\left(\begin{smallmatrix} m & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right) + \tfrac{1}{m+1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

where again, elements of $\mathfrak{pgl}_{m+1}\mathbb{R}$ are equivalent whenever they differ by a scalar multiple of the identity matrix. From this, we can see that

$$\begin{aligned} G_0 := \mathrm{Z}_P(E_{\mathrm{gr}}) &= \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right) \in \mathrm{PGL}_{m+1}\,\mathbb{R} : a \in \mathbb{R}^\times,\ A \in \mathrm{GL}_m\,\mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} \frac{1}{\det(S)} & 0 \\ 0 & S \end{pmatrix} \in \mathrm{PGL}_{m+1}\,\mathbb{R} : S \in \mathrm{GL}_m\,\mathbb{R} \right\} \end{aligned}$$

is the neutral subgroup.

Momentarily, we will also be interested in a specific normal subgroup $G_0^{\mathrm{ss}} \trianglelefteq G_0$, the **s**emi**s**imple part of G_0, given by

$$G_0^{\mathrm{ss}} := \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in \mathrm{PGL}_{m+1}\,\mathbb{R} : A \in \mathrm{SL}^{\pm}_m\,\mathbb{R} \right\},$$

where $\mathrm{SL}^{\pm}_m\,\mathbb{R}$ is the Lie group of linear transformations of $\mathbb{R}^m$ with determinant either $+1$ or -1. This subgroup G_0^{ss} is normal in G_0, and moreover, G_0 decomposes as $G_0 = \exp(\mathbb{R}E_{\mathrm{gr}})G_0^{\mathrm{ss}}$, where $\exp(\mathbb{R}E_{\mathrm{gr}})$ is the image of the one-parameter subgroup generated by E_{gr}. In particular, $G_0/G_0^{\mathrm{ss}} \simeq P/(G_0^{\mathrm{ss}}P_+) \simeq \exp(\mathbb{R}E_{\mathrm{gr}})$.

3. The pedestrian perspective

As before, we'd like to think of our model group $\mathrm{PGL}_{m+1}\,\mathbb{R}$ as the space of configurations for ourselves as pedestrians[1] inside of the geometry of our model. To do this, let's return to the idea of walking around inside a painting.

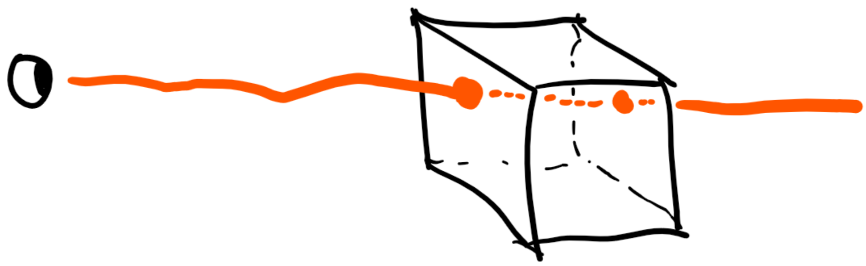

Figure 3. We can reconstruct the $(m+1)$-dimensional scene in an m-dimensional painting by undoing the identification of each sight-line with a point; the result is an embedding of the $(m+1)$-dimensional scene into the principal $\exp(\mathbb{R}E_{\mathrm{gr}})$-bundle $\mathrm{PGL}_{m+1}\,\mathbb{R}/G_0^{\mathrm{ss}}P_+$ over $\mathrm{PGL}_{m+1}\,\mathbb{R}/P \cong \mathbb{RP}^m$

Paintings don't typically depict what is going on both in front and behind the painter, so it makes sense to assume the scene in $\mathbb{R}^{m+1}$ depicted in an m-dimensional painting takes place in some half-space $\{x \in \mathbb{R}^{m+1} : \alpha(x) > 0\}$ for some $\alpha \in (\mathbb{R}^{m+1})^\vee$. Every sight-line for the scene intersects this half-space in a ray, so if we wanted to recreate the

[1]Note that I'm returning to the term "pedestrian" rather than "observer" here, since the "observer perspective" could easily be confused with the perspective of the eye viewing the sight-lines.

$(m+1)$-dimensional scene, then we could just imagine it as occurring in the $\mathbb{R}_+$-bundle over $\mathrm{PGL}_{m+1}\mathbb{R}/P \cong \mathbb{RP}^m$ given by undoing the identification of the ray with a point. This $\mathbb{R}_+$-bundle corresponds to the quotient by $\{\pm 1\}$ of the canonical $\mathbb{R}^\times$-bundle $\mathbb{R}^{m+1} \setminus \{0\}$ over $\mathbb{RP}^m = (\mathbb{R}^{m+1} \setminus \{0\})/\mathbb{R}^\times$; since half-spaces embed into this $\mathbb{R}_+$-bundle by inclusion into $\mathbb{R}^{m+1} \setminus \{0\}$, we lose nothing by assuming the scene happens in this bundle.

Again, $\mathrm{PGL}_{m+1}\mathbb{R}$ acts transitively on the space of sight-lines. The subgroup of $\mathrm{GL}_{m+1}\mathbb{R}$ fixing the sight-line through $\left[\begin{smallmatrix} 1 & 0 & \cdots & 0 \end{smallmatrix}\right]^\top$ *pointwise* is $\left\{\left[\begin{smallmatrix} 1 & \alpha \\ 0 & A \end{smallmatrix}\right] : \alpha^\top \in \mathbb{R}^m,\ A \in \mathrm{GL}_m\mathbb{R}\right\}$, and under the quotient by $\mathbb{R}^\times\mathbb{1}$, the image of this subgroup in $\mathrm{PGL}_{m+1}\mathbb{R}$ is precisely $G_0^{\mathrm{ss}}P_+$. Thus, the $\mathbb{R}_+$-bundle over $\mathbb{RP}^m$ given by undoing the identification of sight-lines with points on the canvas is precisely the principal $\exp(\mathbb{R}E_{\mathrm{gr}})$-bundle $\mathrm{PGL}_{m+1}\mathbb{R}/G_0^{\mathrm{ss}}P_+$ over $\mathrm{PGL}_{m+1}\mathbb{R}/P \cong \mathbb{RP}^m$.

In other words, we imagine the $(m+1)$-dimensional scene depicted in an m-dimensional painting as occurring within the space of the principal $\exp(\mathbb{R}E_{\mathrm{gr}})$-bundle $\mathrm{PGL}_{m+1}\mathbb{R}/G_0^{\mathrm{ss}}P_+$, with right-translation by $\exp(tE_{\mathrm{gr}})$ corresponds to moving closer to the eye if $t < 0$ and farther away if $t > 0$. Note that, from the perspective of the eye, we look smaller when we move farther away and bigger when we move closer.

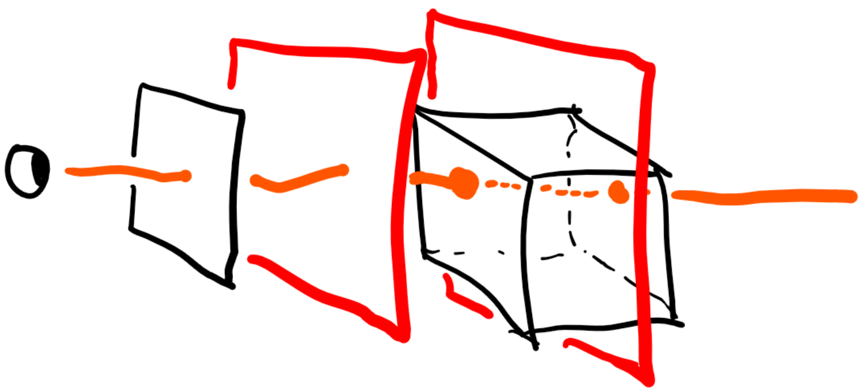

FIGURE 4. Knowing the positioning of the canvas gives us a family of affine hyperplanes in $\mathrm{PGL}_{m+1}\mathbb{R}/G_0^{\mathrm{ss}}P_+$, and our configuration along one of these hyperplanes corresponds to an element of the principal G_0-bundle $\mathrm{PGL}_{m+1}\mathbb{R}/P_+$ over $\mathrm{PGL}_{m+1}\mathbb{R}/P$

By knowing the position of the canvas, we can also describe a family of affine hyperplanes in $\mathrm{PGL}_{m+1}\mathbb{R}/G_0^{\mathrm{ss}}P_+$ that we imagine to be parallel to the canvas (and, in particular, transverse to the sight-line through each point of the scene). Through each point of the scene is one of these affine hyperplanes, and thinking of ourselves as pedestrians within the scene, we can configure ourselves along the affine hyperplane through our point. The choice of configuration gives an element of the principal G_0^{ss}-bundle $\mathrm{PGL}_{m+1}\mathbb{R}/P_+$ over $\mathrm{PGL}_{m+1}\mathbb{R}/G_0^{\mathrm{ss}}P_+$, the space where the scene takes place; this space $\mathrm{PGL}_{m+1}\,\mathbb{R}/P_+$ is then also a principal G_0-bundle over $\mathrm{PGL}_{m+1}\,\mathbb{R}/P$.

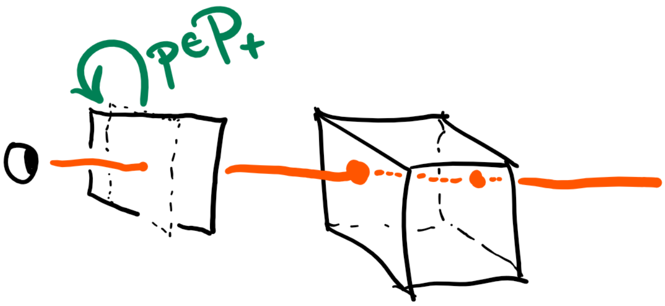

FIGURE 5. Right-translating by an element of P_+ in $\mathrm{PGL}_{m+1}\,\mathbb{R}$ amounts to tilting the choice of affine hyperplane through a point in the scene

Finally, different ways of positioning the canvas result in different families of affine hyperplanes. Given an initial choice of hyperplane through a given point, though, all of the other choices of hyperplane can be obtained by "tilting" the initial one. The "unipotent tilts" $p \in P_+$ run through all of the different choices of hyperplane transverse to the sight-line, so the space of choices of hyperplane is the principal P_+-bundle $\mathrm{PGL}_{m+1}\,\mathbb{R}$ over $\mathrm{PGL}_{m+1}\,\mathbb{R}/P_+$.

Thus, we have arrived at the principal P-bundle $\mathrm{PGL}_{m+1}\,\mathbb{R}$ over $\mathrm{PGL}_{m+1}\,\mathbb{R}/P$. Having built it up from these smaller bundles, we have a fairly good picture of what a configuration within $\mathrm{PGL}_{m+1}\,\mathbb{R}$ looks like, and from this, it's not hard to see how motion works in this case.

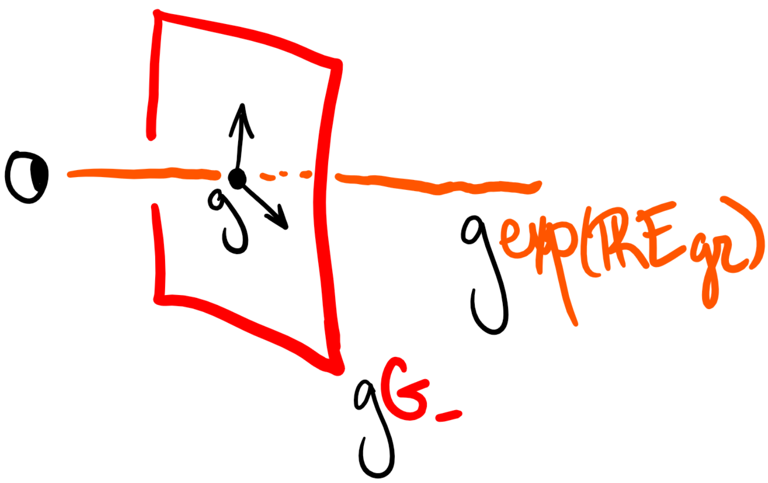

FIGURE 6. Right-translating by elements of G_- amounts to translation within the affine hyperplane through the point in the scene, while right-translating by an element of $\exp(\mathbb{R}E_{\mathrm{gr}})$ amounts to moving along the sight-line

An element $g \in \mathrm{PGL}_{m+1}\mathbb{R}$ determines a choice of affine hyperplane within the $(m+1)$-dimensional scene of an m-dimensional painting, and as we might guess from last time, translation along this affine hyperplane amounts to right-translation by elements of G_-. Then, right-translating by elements of G_0^{ss} corresponds to changing our frame within this affine hyperplane, while right-translating by an element of $\exp(\mathbb{R}E_{\mathrm{gr}})$ amounts to moving along the sight-line through our point in the scene; from the perspective of the eye, right-translation by an element of $\exp(\mathbb{R}E_{\mathrm{gr}})$ also corresponds to rescaling the affine hyperplane. Finally, right-translating by an element of P_+ tilts our choice of affine hyperplane.

4. Geodesics

The choice of affine hyperplane described above corresponds to a choice of affine patch in projective space. Inside of a given affine patch are affine geodesics, corresponding to the images $t \mapsto q_P(g\exp(tv))$ for $v \in \mathfrak{g}_-$. However, the images of these aren't going to be full copies of geodesics on $\mathbb{RP}^m$ in the sense we'd usually mean, since they are restricted to an affine patch.

A full (unparametrized) geodesic in $\mathbb{RP}^m$ corresponds to a choice of (two-dimensional) plane through the origin inside of $\mathbb{R}^{m+1}$. More specifically, thinking of $\mathbb{RP}^m$ as the space of one-dimensional subspaces of $\mathbb{R}^{m+1}$, a geodesic is the set of all one-dimensional subspaces lying in a given two-dimensional subspace. This is analogous, and in fact related, to the situation in spherical geometry, where we defined great circles to be intersections of the unit sphere with two-dimensional subspaces.

Note, as we did with spherical geometry, that such a definition is geometric for the model: elements of $\mathbb{R}^\times\mathbb{1}$ preserve every subspace of $\mathbb{R}^{m+1}$, and elements of $\mathrm{GL}_{m+1}\mathbb{R}$, being invertible linear transformations, send two-dimensional subspaces to two-dimensional subspaces, so $\mathrm{PGL}_{m+1}\mathbb{R}$ sends two-dimensional subspaces to two-dimensional subspaces.

Conveniently, the affine geodesics inside a given affine patch are the intersections of full geodesics with that affine patch. Indeed, identifying $\mathrm{PGL}_{m+1}\mathbb{R}/P$ with $\mathbb{RP}^m$,

$$q_P\Big(g\exp\left(t\left(\begin{smallmatrix} 0 & 0 \\ v & 0 \end{smallmatrix}\right)\right)\Big) = g \cdot q_P\Big(\left(\begin{smallmatrix} 1 & 0 \\ tv & \mathbb{1} \end{smallmatrix}\right)\Big) \in \left\langle g \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g \cdot \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right)\right\rangle .$$

This situation is noteworthy; in other parabolic model geometries, the geodesics corresponding to one-parameter subgroups generated by elements of $\mathfrak{g}_-$ might not correspond to particularly meaningful curves in the base manifold at all. The corresponding motion in the model group will always be geometrically meaningful though, so it is still worthwhile if we view things from our observer perspective.